Brownian motion and Stochastic Calculus
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Assignment 12

Exercise 1

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion in $[0,T]$ and a_1, a_2, b_1, b_2 deterministic functions of time. The general form of a scalar *linear stochastic differential equation* is

$$\mathrm{d}X_t = \big(a_1(t)X_t + a_2(t)\big)\mathrm{d}t + \big(b_1(t)X_t + b_2(t)\big)\mathrm{d}B_t. \tag{0.1}$$

If the coefficients are measurable and bounded on $[0,T]$, we can apply our general result to get existence and uniqueness of a strong solution $(X_t)_{t\in[0,T]}$ for each initial condition x.

1) When $a_2(t)\equiv 0$ and $b_2(t)\equiv 0$, (0.1) reduces to the *homogeneous linear SDE*

$$\mathrm{d}X_t = a_1(t)X_t\mathrm{d}t + b_1(t)X_t\mathrm{d}B_t. \tag{0.2}$$

Show that the solution of (0.2) with initial data $x=1$ is given by

$$X_t = \exp\bigg(\int_0^t \Big(a_1(s) - \frac{1}{2}b_1^2(s)\Big)\mathrm{d}s + \int_0^t b_1(s)\mathrm{d}B_s\bigg).$$

2) Find a solution of the SDE (0.1) with initial condition $X_0 = x$.

3) Solve the *Langevin's SDE*

$$\mathrm{d}X_t = a(t)X_t\mathrm{d}t + \mathrm{d}B_t,\ X_0 = x.$$

Exercise 2

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space carrying a one-dimensional Brownian motion B, whose $\mathbb{P}$-augmented filtration is denoted by $\mathbb{F}$. Fix positive constants T and γ, and let ξ be a bounded $\mathcal{F}_T$-measurable random variable.

1) Show that the process

$$Y_t := -\gamma\log\Big(\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\xi/\gamma}\big|\mathcal{F}_t\big]\Big),\ t\geq 0,$$

is the first component of a solution to the BSDE with terminal condition ξ (at T) and generator g with

$$g(z) := -\frac{1}{2\gamma}z^2,\ z\in\mathbb{R}.$$

2) Let $b\in\mathbb{R}$. Show that the process

$$Y_t := -\gamma\bigg(\frac{b^2}{2}(T-t) - bB_t + \log\Big(\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{bB_T-\xi/\gamma}\big|\mathcal{F}_t\big]\Big)\bigg),\ t\geq 0,$$

is the first component of a solution to the BSDE with terminal condition ξ (at T) and generator g with

$$g(z) := -\frac{1}{2\gamma}z^2 - bz,\ z\in\mathbb{R}.$$

Exercise 3

Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X_t)_{t\geq 0}$ the unique solution of the SDE

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}B_t,\ X_0 = x,$$

where $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz-continuous functions.

1) Find a non-constant function $\phi(x) \in C^2(\mathbb{R}, \mathbb{R})$ such that $Y_t := \phi(X_t)$ is a local martingale. Moreover, derive a SDE for $(Y_t)_{t\geq 0}$.

   Hint: Prove and use that general solution of the ODE: $y' f(x) + \frac{1}{2} y'' g^2(x) = 0$ is of the form

$$y(x) = a + b\int_0^x \exp\bigg(-2\int_0^u \frac{f(v)}{g^2(v)}\mathrm{d}v\bigg)\mathrm{d}u,\ (a,b) \in \mathbb{R}^2.$$

2) Assume additionally that f is negative on $(-\infty, 0)$ and positive on $[0, \infty)$. Show that in this case, Y is a martingale.

Exercise 4

1) Let $(f_t)_{t\geq 0}$ be an $\mathbb{F}$-adapted, positive, increasing, differentiable process starting from zero and consider the following SDE

$$\mathrm{d}X_t = \sqrt{f_t'}\mathrm{d}B_t. \tag{0.3}$$

   Show that the process B_{f_t} is a weak solution of (0.3).

   Hint: in other words, given a Brownian motion $(B_t)_{t\geq 0}$ and a function f satisfying the previous assumptions, there exists a Brownian motion $(\widehat{B}_t)_{t\geq 0}$, such that

$$\mathrm{d}\widehat{B}_{f_t} = \sqrt{f_t'}\mathrm{d}B_t.$$

2) Recall that a solution of the SDE

$$\mathrm{d}X_t = -\gamma X_t \mathrm{d}t + \sigma \mathrm{d}B_t,\ X_0 = x, \tag{0.4}$$

   is called Ornstein–Uhlenbeck process. Show that an Ornstein–Uhlenbeck process has representation

$$X_t = \mathrm{e}^{-\gamma t}\widetilde{B}_{\psi(t)},$$

   where

$$\psi(t) := \frac{\sigma^2(\mathrm{e}^{2\gamma t} - 1)}{2\gamma},$$

   and where $(\widetilde{B}_t)_{t\geq 0}$ is a Brownian motion started at x.

3) Consider the SDE

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t,\ X_0 = x, \tag{0.5}$$

   with $\sigma(x) > 0$ such that

$$G(t) := \int_0^t \frac{\mathrm{d}s}{\sigma^2(B_s)},$$

   is finite for finite t, and increases to infinity, that is $G(\infty) = \infty$, $\mathbb{P}$–a.s. Under these assumptions, the inverse of G is well-defined, and we let

$$\tau_t := G_t^{(-1)}.$$

   Show that the process $X_t := B_{\tau_t}$ is a weak solution to the SDE (0.5).